# Flat families

**Recall**: An $R$-module $M$ is <u>flat</u> if $\forall$ $N' \subseteq N$ $R$-modules, the induced map $M \otimes_R N' \longrightarrow M \otimes N$ is injective. (Recall that tensoring always preserves surjections.)

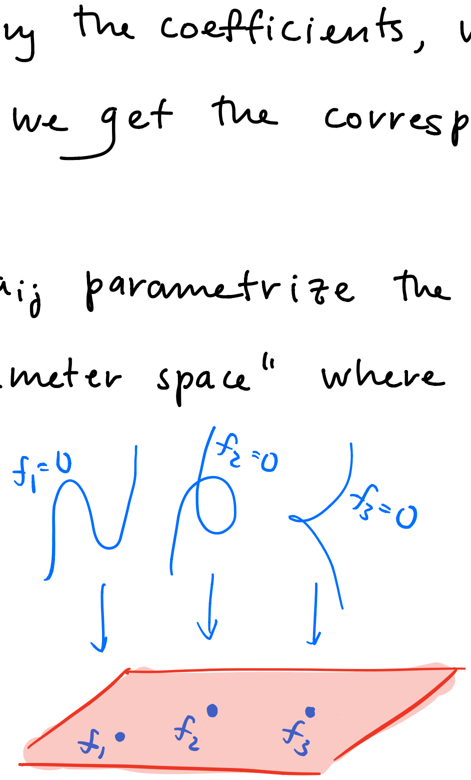
We showed that localization is exact. i.e. $R[u^{-1}]$ is a flat $R$-module.

We will see that this notion is closely connected to the idea of varieties/schemes or algebras varying in a family.

**Ex**: The most natural example of a family is plane curves of degree $d$. They are given by $f = \sum_{i+j\leq d} a_{ij}\, x^i y^j$.

When we vary the coefficients, we vary the curve. Algebraically, we get the corresponding algebras $k[x,y]/(f)$.

Roughly, the $a_{ij}$ parametrize the curves and we get a map to the "parameter space" where each fiber is the corresponding curve.

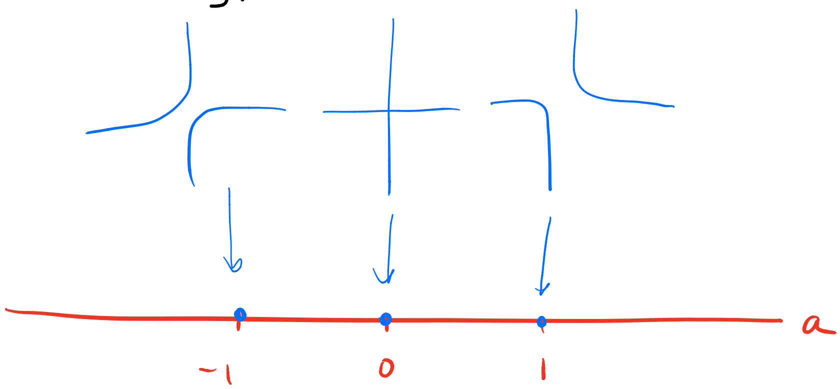


So why not just define a family as a morphism?

i.e. each member of the family is a fiber.

This definition is too general. In the curve example, varying the parameters also varies the geometry of the curve in a natural way.

e.g. $1xy - a = 0$ gives a (sub)family of curves, and as $a \to 0$, it deforms from a hyperbola into the union of two lines




However, if all the coefficients are 0, $f=0$, so we get the whole plane, which doesn't fit in nicely in a family of curves.

Even worse, if $B \to \mathbb{A}^2$ is the blowup of $\mathbb{A}^2$ at the origin, all the fibers are single points other than the exceptional fiber, which is a whole line.

It turns out that the way we exclude these cases is by requiring the map to be flat.

i.e. for a morphism of varieties $\varphi: X \to B$, and $x \in X$,

we want there to be an open neighborhood

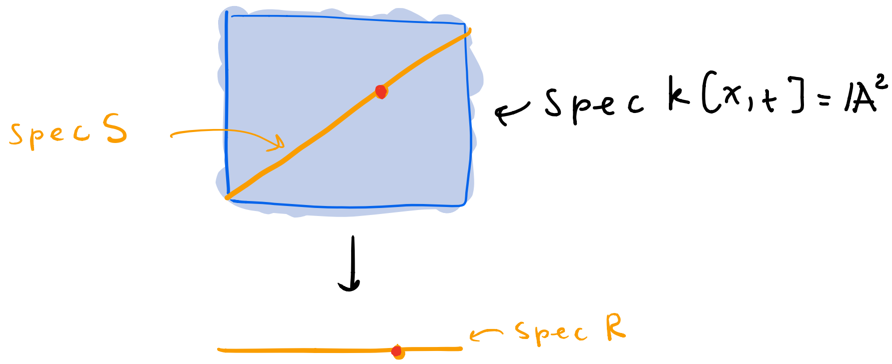
$U$ of $x$ and $V$ of $\varphi(x)$ s.t. $\varphi$ restricts to $U \to V$, and the corresponding map of rings $R \to S$ makes $S$ flat as an $R$-module.

We'll first give some more geometric examples, and then investigate flatness more deeply.

For these examples, set $R = k[t]$, $k = \bar{k}$.

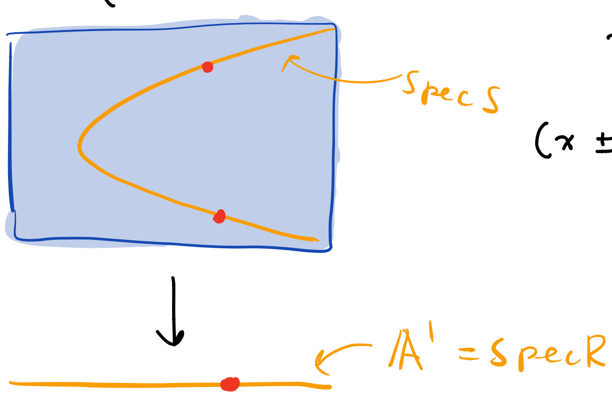
<u>Ex</u>: $S = R[x]/(x-t)$. $S \cong R$, and $R \otimes_R N = N$ $\forall$ $N$, so $R$ is flat.

Geometrically, we have the following




Over the point $(t-a)$ in Spec $R$, we have $(x-a, t-a)$

<u>Ex</u>: $S = R[x]/(x^2-t)$




Preimage of $(t-a)$ is $(x \pm a, t-a) = \operatorname{Spec}\left(k[x]/(x^2-a)\right) = 2$ points for $a \neq 0$.

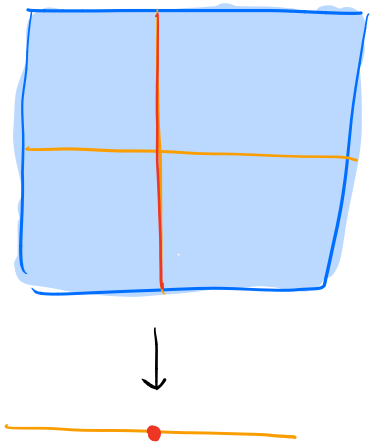
Preimage of $(t)$ is $\operatorname{Spec}\left(k[x]/(x^2)\right) = 1$ point

However, notice that $\dim_k\left(k[x]/(x^2-a)\right) = 2$ $\forall$ $a$.

$S$ is a free $R$-module of rank 2 ($x^2 - t$ monic),

so $S \otimes_R N \cong R^{\oplus 2} \otimes_R N = N \oplus N$ for all $R$-modules $N$. In particular, tensoring by $S$ preserves injections, so it's flat.

**Ex:** $S = R[x]/(t(x-1))$. In $S$, $tx = t^2$



The fiber over $(t-a)$, $a \neq 0$, is $\operatorname{Spec}\left(k[x]/(x-1)\right)$ = one point

If $a = 0$, the fiber is $\operatorname{Spec} k[x] = \mathbb{A}^1$

So the fibers don't vary nicely in a family. In fact, we will soon see that $S$ is <u>not</u> flat over $R$.

## Free resolutions and Tor

When discussing flatness, it's very useful to know a little homological algebra. Here's a brief intro to Tor:

Let $M$ be an $R$-module. A <u>free resolution</u> of $M$ is a sequence

$$\cdots \to F_1 \to F_0 \to M \to 0$$

where each $F_i$ is a free module, and the sequence is exact.

Note that we can construct a free resolution for any module as follows:

Let $\{m_i\}_{i\in J}$ be a generating set for $M$. Then set $F_0 = \bigoplus_{i\in J} R$, and let $e_i$ be the $i^{\text{th}}$ basis vector in $F_0$.

We have $\begin{array}{c} F_0 \longrightarrow M \\ e_i \longmapsto m_i \end{array}$ surjective. If $M_0 = \ker(F_0 \to M)$, we repeat this process w/ $M_0$ and get $F_1 \to F_0 \to M \to 0$ exact.

We can continue this process (possibly eventually reaching $0$).

Ex: let $M = k[x,y]/(x,y) \cong k$. This is generated by $1$.

$$\begin{array}{ccccc} R & \longrightarrow & M & \longrightarrow & 0 \\ 1 & \longmapsto & 1 & & \end{array}$$

The kernel is $(x,y)$, which is generated by $x$ and $y$.

$$\begin{array}{ccccccc} R^2 & \longrightarrow & R & \longrightarrow & M & \longrightarrow & 0 \\ (a,b) & \longmapsto & ax+by & & & & \end{array}$$

The kernel is generated by $(-y,x)$.

$$\begin{array}{ccccccccccc} 0 & \to & R & \longrightarrow & R^2 & \longrightarrow & R & \longrightarrow & M & \longrightarrow & 0 \\ & & f & \longmapsto & (-fy, fx) & & & & & & \end{array}$$

injective.

This gives us a free resolution. Note that this is not unique. It requires a choice of generators, and we could also add on superfluous $R$ summands at any point and just send them to $0$.

**Def:** Let $M$ and $N$ be $R$-modules, and

$$\cdots \to F_1 \to F_0 \to M \to 0$$

a free resolution of $M$. $\mathrm{Tor}_i^R(M,N)$ is the homology at $F_i \otimes N$ of the complex

$$\cdots \to F_{i+1} \otimes N \to F_i \otimes N \to F_{i-1} \otimes N \to \cdots \to F_0 \otimes N \to 0$$

leave out $M$

That is, it is $\dfrac{\ker(F_i \otimes N \to F_{i-1} \otimes N)}{\mathrm{im}(F_{i+1} \otimes N \to F_i \otimes N)}$

## Facts about Tor:

1.) It's well-defined. i.e. it's independent of the chosen resolution.

2.) $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}_i(N,M)$, i.e. we can compute it by finding a free resolution of $N$ instead.

3.) $\mathrm{Tor}_0^R(M,N) = \mathrm{coker}(M_1 \otimes N \to M_0 \otimes N)$. But tensoring is right exact, so this is just $M \otimes N$.

4.) If $M$ is free, then $0 \to M \to M \to 0$ is a free resolution.

$\Longrightarrow \mathrm{Tor}_i(M,N) = 0 \ \forall i > 0.$

5.) $\mathrm{Tor}_i^R(M,N)$ is $R$-bilinear: mult. on $M$ by $r \in R$ induces mult. by $r$ on $\mathrm{Tor}_i^R(M,N)$

6.) $R$ Noetherian, $M$ and $N$ f.g. $\Rightarrow \mathrm{Tor}_i^R(M,N)$ f.g.

7.) If $S$ is a flat $R$-algebra, then

$$S \otimes_R \operatorname{Tor}_i^R(M,N) = \operatorname{Tor}_i^S(S\otimes_R M,\ S\otimes_R N). \quad \text{(Exercise)}$$

8.) Tor is the left derived functor of tensor product. That is, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $R$-modules, then there is a long exact sequence

$$\cdots \to \operatorname{Tor}_2(M'',N) \to \operatorname{Tor}_1(M',N) \to \operatorname{Tor}_1(M,N) \to \operatorname{Tor}_1(M'',N) \to M'\otimes N \to M\otimes N \to M''\otimes N \to 0$$

**Ex:** Let $R = k[x]$. Then we have the following free resolution of $k[x]/(x)$:

$$0 \to R \xrightarrow[\cdot x]{} R \to R/(x) \to 0$$

If $M$ is any $R$-module, we have

$$\operatorname{Tor}_i(R/(x), M) = H_i(0 \to M \xrightarrow{\cdot x} M \to 0), \text{ so}$$

$$\operatorname{Tor}_0(R/(x), M) = M/xM$$

$$\operatorname{Tor}_1(R/(x), M) = \{m \in M \mid xm = 0\}$$

and all higher $\operatorname{Tor}_i$ are $0$.

In fact this holds more generally for any $R$, and $x \in R$ a NZD.

## Tor and flatness

There is an immediate connection between Tor and flatness:

$$\operatorname{Tor}_1^R(M,N) = 0 \Rightarrow M \text{ is flat} \Rightarrow \operatorname{Tor}_i^R(M,N) = 0 \ \forall\, i > 0.$$

(see next HW)

Of course, this can be hard to check, but we have the following stronger criterion for flatness:

**Prop:** $R$ a ring, $M$ an $R$-module.

1.) If $I \subseteq R$ is an ideal, $I \otimes_R M \to M$ (the multiplication map) is an injection $\iff \operatorname{Tor}_1^R(R/I, M) = 0$.

2.) $M$ is flat $\iff$ the condition in 1.) is satisfied $\forall$ ideals $I \subseteq R$.

**Pf:** 1.) Consider the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

Then we get a long exact sequence:

$$\cdots \to \underset{\substack{\parallel \\ 0}}{\operatorname{Tor}_1(R,M)} \to \operatorname{Tor}_1(R/I, M) \to I \otimes M \to \underset{\substack{\parallel \\ M}}{R \otimes M} \to \cdots$$

so this is $0$ iff is injective.

2.) If $M$ is flat, the condition is satisfied by definition.

Assume the condition is satisfied $\forall$ ideals $I \subseteq R$.

Let $N' \subseteq N$ be $R$-modules.

Consider the map $N' \otimes M \to N \otimes M$.

This condition only involves finitely many elements of $N$.
Replace $N$ w/ the submodule generated by those elements.
Then we can assume $N$ is finitely generated, by $n$ generators.

Then $N/N'$ is finitely generated.

Let $N' = N_0 \subset N_1 \subset \ldots \subset N_p = N$, where $N_{i+1}/N_i$ is generated by one element.

If $N_i \otimes M \to N_{i+1} \otimes M$ is injective for each $i$, we're done, so assume $N/N'$ is generated by one element. Then $R \twoheadrightarrow N/N'$, so $N/N' \cong R/I$, some ideal $I \subseteq R$.

$$\underbrace{\mathrm{Tor}_1(N/N', M)}_{0} \to N' \otimes M \to N \otimes M \quad \text{is exact.}$$

0 by assumption, so $N' \otimes M \to N \otimes M$ is injective. □

In fact, with a little tweaking, one can show that the condition only needs to hold for finitely generated ideals.

From this, we get a nice special case about modules over PIDs.
First we need a definition:

**Def:** An $R$-module $M$ is torsion free if $r \in R$ a NZD, $m \in M$ nonzero then $rm \neq 0$. i.e. the only element in $M$ annihilated by a nonzero divisor in $R$ is $0$.

**Cor:** 1.) If $M$ is a flat $R$-module, then $M$ is torsion free.

2.) If $R$ is a PID, then $M$ is flat iff $M$ is torsion free.

**Pf:** 1.) Let $a \in R$ be a nonzerodivisor (in $R$).

Define $R \to R$ by $r \mapsto ar$. This is an injection. Since $M$ is flat, $R \otimes M \xrightarrow[\cdot a]{} R \otimes M$ is an injection. But this is the map $M \xrightarrow{\cdot a} M$, so $a$ is a NZD on $M$, so $M$ is torsion free.

2.) "$\Rightarrow$" holds by 1. For "$\Leftarrow$", assume $M$ is torsion free over $R$, a PID (and thus a domain).

Let $I \subseteq R$ be an ideal. Then $I = (a)$. WTS $\mathrm{Tor}_1^R(R/(a), M) = 0$.

If $a = 0$, we're done since $R$ is flat. If $a \neq 0$, then by an example above,

$\mathrm{Tor}_1(R/(a), M) = \{m \in M \mid am = 0\} = 0$. $\square$

Now we return to the example from before:

**Ex:** $k = \bar{k}$, $R = k[t]$, $S = R[x]/(t(x-1))$. $S$ has torsion: $t(x-1) = 0$, so $S$ is not flat over $R$.

A really important property of flatness is that it is local. Geometrically, that means that you can check for flatness in an infinitesimal neighborhood of each point. Algebraically, this means we can check flatness by localizing at each prime:

**Prop:** $M$ is flat over $R \iff M_p$ is flat over $R_p$ for all primes $P$.

**Pf:** Suppose $M$ is flat over $R$. Then if

$$0 \to N' \to N \to N'' \to 0$$ is an exact sequence of $R_p$-modules,

$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$ is exact too.

But $M \otimes_R N \cong M_p \otimes_{R_p} N$ as $R_p$-modules:

$$m \otimes n/u \longmapsto m/1 \otimes n/u$$ and the inverse is $$m/v \otimes n/u = m/1 \otimes n/uv \mapsto m \otimes n/uv.$$

Thus, $M_p$ is flat.

Now assume $M$ is <u>not</u> flat. Then there's some short exact sequence of $R$-modules $0 \to N' \to N \to N'' \to 0$ that's not exact when tensoring by $M$. Then

$$0 \to K \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$ is exact.

kernel of

$K \neq 0$, so there is some prime $P$ s.t. $K_p \neq 0$. Thus

$$0 \to K_p \to M_p \otimes_{R_p} N'_p \to M_p \otimes_{R_p} N_p \to M_p \otimes_{R_p} N''_p \to 0$$ is exact, so $M_p$ isn't flat. $\square$

Note that we could've replaced "prime ideals" with "maximal ideals".

**Ex:** The "problem point" of $M = k[x,t]/(t(x-1))$ as an $R = k[t]$-module was $(t)$. In fact, localizing at $t$, we get

$$M_{(t)} = k[x,t]_{(t)}/(t(x-1))_{(t)}$$ which has torsion.

At any other point, $t$ becomes a unit, so

$$M_{(t-a)} = k[x,t]_{(t-a)}/(x-1) \cong k[t]_{(t-a)},$$ which is a free $R_{(t-a)}$-module

and thus flat.